and $\vec{\nabla}_v$ represents differentiate w.r.t v_x, v_y, v_z.

therefore —

$$\frac{f(\vec{r},\vec{v},t+\delta t) - f(\vec{r},\vec{v},t)}{\delta t} = -\vec{v}.\vec{\nabla}f - \vec{a}\,\vec{\nabla}_v f \quad \text{—— (6)}$$

taking limit $\delta t \to 0$ both sides we have

$$\frac{\partial f}{\partial t} + \vec{v}.\vec{\nabla}f + \vec{a}.\vec{\nabla}_v f = 0 \quad \text{—— (7)}$$

This is the Boltzmann eqn in absence of collision.

If the collisions of the particles is taken into the acount the Boltzmann equations becomes

$$\frac{\partial f}{\partial t} + \vec{v}.\vec{\nabla}f + \vec{a}.\vec{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_c \quad \text{—— (8)}$$

The collision term $\left(\frac{\partial f}{\partial t}\right)_c$ on the R.H.S of (8) should be properly evaluated from the collision dynamics of the interacting particles. It depends on nature of the particles and also the nature of the collisions.

Vlasov equation:-

The Vlasov equation is a partial differential equation that describes the time evolution of the distribution func $f(\vec{r},\vec{v},t)$ in phase space. It can be obtained from the Boltzmann eqn (8) with collision term neglected but including the self-consistent internal electromagnetic fields produced by the presence and motion of all charge particle inside the plasma. The Vlasov eqn is given by

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{q}{m}\underbrace{\left(\vec{E} + \vec{v}\times\vec{B}\right)}_{\vec{a}}\cdot\vec{\nabla}_v f = 0$$

Sometimes the Vlasov eqn is referred to as the collisionless Boltzmann eqn. Because of its comparative simplicity Vlasov eqn is widely used to describe the dynamics of plasma, particularly sufficiently hot plasma where collision can be neglected.

Equation of Continuity:- using Vlasov eqn

let us consider the zeroth velocity moment of Boltzmann eqns or Vlasov eqns which is given by

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right)\cdot\vec{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_c \quad \text{—(i)}$$

Since $v^0 = 1$ the zeroth order moment implies a simple integration of each term of (i) over the velocity space.

Thus we get —

$$\int_v \frac{\partial f}{\partial t} d^3v + \int_v \vec{v}\cdot\vec{\nabla}f\, d^3v + \frac{q}{m}\int_v (\vec{E}+\vec{v}\times\vec{B})\cdot\vec{\nabla}_v f\, d^3v = \int_v \left(\frac{\partial f}{\partial t}\right)_c d^3v$$

— (ii)

The 1st term can be express as

$$\int_v \left(\frac{\partial f}{\partial t}\right) d^3v = \frac{\partial}{\partial t}\int_v f\, d^3v = \frac{\partial n}{\partial t} \quad — (3)$$

Since $\vec{r}, \vec{v}, t$ are independent variables

the 2nd term

$$\int_v \vec{v}\cdot\vec{\nabla}f\, d^3v = \vec{\nabla}\cdot\int_v \vec{v}f\, d^3v = \vec{\nabla}\cdot(n\vec{u})$$

— (4)

The 3rd term can be express as

$$\int_v \vec{a}\cdot\vec{\nabla}_v f\, d^3v = \int_v \vec{\nabla}_v\cdot f\vec{a}\, d^3v - \int f\,\vec{\nabla}_v\cdot\vec{a}\, d^3v$$

$$= \int_{S_\infty} f\vec{a}\cdot d\vec{S} - \int_v f\,\vec{\nabla}_v\cdot\vec{a}\, d^3v$$

— (5)

[using Gauss's div theorem]

$$\left[\vec{a} = \frac{q}{m}(\vec{E}+\vec{v}\times\vec{B})\right]$$

where S_∞ is the v-space surface at $v \to \infty$, As f vanishes faster than v^2 and area in velocity space increases as v^2 the 1st integral vanishes in (5) for any distribution with finite energy. The 2nd integral R.H.S of

(5) also vanishes if we assume

$$\vec{\nabla}_v \cdot \vec{a} = \vec{\nabla}_v \cdot \frac{\vec{F}}{m} = 0$$

This is done if the force component $F_i \, (i = x, y, z)$ is independent of the velocity component v_i. It is also done for both the electromagnetic forces that is for $\vec{E}$ and $\vec{v} \times \vec{B}$

The fourth term in eqn (2) represents the rate per unit volume at which particles are produced or lost as a result of collisions. So in the absence of interaction we must have $\left(\frac{\partial f}{\partial t}\right)_c = 0$ so fourth term becomes zero.

Using the above results eqn (1) becomes

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{u}) = 0$$ which is the required eqn of continuity.

Equation of motion :-

Let us consider the ~~zeroth velocity moment~~ ~~of~~ Boltzmann eqns or Vlasov equations which is given by —

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{m} (\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{\nabla}_v f = \left(\frac{\partial f}{\partial t}\right)_c \quad \text{--- (1)}$$

Now let us ~~take~~ multiply (1) $m\vec{v}$ [i.e. the 1st order velocity moment] and integrating (1) over the velocity space

$$m \int_v \vec{v} \frac{\partial f}{\partial t} d^3v + m \int_v \vec{v} (\vec{v} \cdot \vec{\nabla} f) d^3v$$

$$+ m \int_v \vec{v} (\vec{a} \cdot \vec{\nabla}_v) f \, d^3v = \int_v m\vec{v} \left(\frac{\partial f}{\partial t}\right)_c d^3v \quad \text{--- (2)}$$

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The 1st term may be expressed that

$$m \int_v \vec{v} \frac{\partial f}{\partial t} d^3v = m \frac{\partial}{\partial t} \int_v \vec{v} f \, d^3v$$

$$= m \frac{\partial}{\partial t} (n\vec{U}) \quad \text{(3)}$$

Since $\vec{r}, \vec{v}, t$ are independent variables

the 2nd term —

$$m \int_v \vec{v}(\vec{v} \cdot \vec{\nabla}) f \, d^3v = m \vec{\nabla} \cdot \int_v \vec{v} \vec{v} f \, d^3v$$

$$= m \vec{\nabla} \cdot n \langle \vec{v} \vec{v} \rangle \quad \text{(4)}$$

$$\vec{v} = \vec{u} + \vec{w}$$

$$\vec{w} = \vec{v} - \vec{u} \quad \text{(5)}$$

where w represents the random velocity of a particle relative to the mean fluid velocity.

From vector calculus from eqn (5) we have

$$m \vec{\nabla} \cdot n(\vec{v}\vec{v}) = m \vec{\nabla} \cdot (n\vec{u}\vec{u}) + m\vec{\nabla} \cdot [n\langle\vec{w}\vec{w}\rangle] + m \, 2\vec{\nabla} \cdot [n\vec{u}\langle\vec{w}\rangle] \quad \text{(6)}$$

the 2nd velocity moment of $f(\vec{r}, \vec{v}, t)$ gives the pressure tensor as

$$\vec{P}(\vec{r}, t) = m \int_v (\vec{v} - \vec{u})(\vec{v} - \vec{u}) f \, d^3v \quad \text{(7)}$$

Since $\langle\vec{w}\rangle = 0$ and by defn (7)

the quantity $mn\langle\vec{w}\vec{w}\rangle$ represents the kinetic pressure tensor $\overleftrightarrow{p}$

and also

$m\vec{\nabla}\cdot(n\vec{u}\vec{u}) = m\vec{u}\cdot\vec{\nabla}\cdot(n\vec{u}) + mn(\vec{u}\cdot\vec{\nabla})\vec{u}$

we can write the 2nd term of eqn (1) as

$$m\vec{u}\,\vec{\nabla}\cdot(n\vec{u}) + mn(\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}\cdot\vec{P} \quad (8)$$

the 3rd term equation (1) can be written as

$$m\int_v \vec{\nabla}_v\cdot(\vec{a}\vec{v}f)\,d^3v - m\int_v f\vec{v}\,\vec{\nabla}_v\cdot\vec{a}\,d^3v$$
$$- m\int_v f\vec{a}\cdot\vec{\nabla}_v\vec{v}\,d^3v \quad (9)$$

using Gauss's divergence theorem the 1st integral in (9) can be written as

$$m\int_{S_\infty}(\vec{a}\vec{v}f)\,ds$$

where S_∞ is the v-space surface at $v\to\infty$. for any distribution with finite energy the quantity $\vec{a}\vec{v}f$ vanishes and hence the integral vanishes.

The 2nd integral in (9) vanishes if we assume that $\vec{\nabla}_v\cdot\vec{a} = \vec{\nabla}_v\cdot\frac{\vec{F}}{m} = 0$

This is done is the force component F_i $(i = x, y, z)$ is independent of velocity component v_i. which ~~This~~ is also done for both electromagnetic forces that is $\vec{E}$ and $\vec{v}\times\vec{B}$. Thus 3rd term the eqn (9) becomes

$$- m \int_v f \vec{a} \cdot \vec{\nabla}_v \vec{v} \, d^3v$$

$$= - q \int_v f \left(\vec{E} + \vec{v} \times \vec{B} \right) \cdot \vec{\nabla}_v \vec{v} \, d^3v \quad \text{—— (10)}$$

Since $\vec{\nabla}_v \vec{v}$ is an identity tensor

eqn (10) becomes —

$$- qn \left(\vec{E} + \vec{u} \times \vec{B} \right)$$

The R.H.S of the eqn (2) represents the change of momentum due to collision ~~and~~ denoting it by P_{coll} and using the results (2), (4), (5) then (2) becomes —

$$m \frac{\partial}{\partial t} \left(n \vec{u} \right) + m \vec{u} \, \vec{\nabla} \cdot \left(n \vec{u} \right) + mn \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} + \vec{\nabla} \cdot \vec{P} - qn \left(\vec{E} + \vec{u} \times \vec{B} \right) = P_{coll} \quad \text{—— (11)}$$

$$\Rightarrow mn \left[\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right] \vec{u} = qn \left(\vec{E} + \vec{u} \times \vec{B} \right) - \vec{\nabla} \cdot \vec{P} + P_{coll} \quad \text{—— (12)}$$

which is the required transport momentum equations for a fluid plasma.

rest

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